October Math Gems

Problem of the week 12

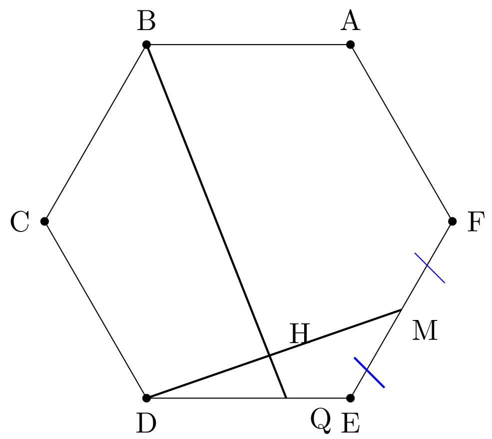
§1 Problems

Problem 1.1. If a, b are real numbers such that $a \geq b$, prove that

$$\left(\frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{8b}\right)^2 \geq \frac{1}{16}.$$

Problem 1.2. We have 6 balls of different colours, including blue and red. In how many ways can we arrange them in a row so that the blue ball and red ball do not come together.

Problem 1.3. ABCDEF is a regular hexagon. Given that M is the midpoint of $\overline{\mathrm{EF}}$ and $\overline{\mathrm{BQ}}$ is perpendicular to $\overline{\mathrm{DM}}$ at H. Find the ratio $\frac{DH}{HM}$.



Problem 1.4. Find all pairs of integers (a, b) such that

$$a^2 + b = 2023.$$

Problem 1.5. Equal letters stand for equal numbers, different letters for different numbers.

$$4 \cdot ABCDE = EDCBA.$$

Determine $ABCDE$.

Problem 1.6. Find all integers satisfying the equation

$$2^x \cdot (4-x) = 2x + 4$$

Problem 1.7. $a_1a_2a_3$ and $a_3a_2a_1$ are two three-digit decimal numbers, with a_1, a_3 being different non-zero digits. The squares of these numbers are five-digit numbers $b_1b_2b_3b_4b_5$ and $b_5b_4b_3b_2b_1$ respectively. Find all such three-digit numbers.

Problem 1.8. Let's call a positive integer "interesting" if it is a product of two (distinct or equal) prime numbers. What is the greatest number of consecutive positive integers all of which are "interesting"?

Problem 1.9. Solve the system of equations:

$$\begin{cases} x^5 = y + y^5 \\ y^5 = z + z^5 \\ z^5 = t + t^5 \\ t^5 = x + x^5 \end{cases}$$

Problem 1.10. Determine all real numbers a, b, c, d that satisfy the following system of equations.

$$\begin{cases} abc + ab + bc + ca + a + b + c = 1 \\ bcd + bc + cd + db + b + c + d = 9 \\ cda + cd + da + ac + c + d + a = 9 \\ dab + da + ab + bd + d + a + b = 9 \end{cases}$$

Problem 1.11. Find all pairs of integers (p, q) such that

$$(p-q)^2 = \frac{4pq}{p+q-1}$$

Problem 1.12. Assume that $g(x) = \frac{x^2}{1+x^2})$. Find the value of the expression

$$g(\frac{1}{2000}) + g(\frac{2}{2000}) + \cdots + g(\frac{1999}{2000}) + g(\frac{2000}{2000}) + g(\frac{2000}{1999}) + \cdots + g(\frac{2000}{1})$$

Problem 1.13. Let a, b, c and d be non-negative integers. Prove that the numbers 2^a7^b and 2^c7^d give the same remainder when divided by 15 if and only if the numbers $3^a5^b$and 3^c5^d give the same remainder when divided by 16.

Problem 1.14. If XYZ is a triangle with Y and Z being acute and different from $\frac{\pi}{4}$, and we also have L as the foot of the height from X, then prove that $\angle X$ is right if and only if

$$\frac{1}{XL - YL} + \frac{1}{XL - ZL} = \frac{1}{XL}$$

Problem 1.15. Given x and y as positive real numbers, where

$$\frac{x^3}{y^2} + \frac{y^3}{x^2} = 5\sqrt{5xy}.$$

Show that

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \sqrt{5}.$$

Problem 1.16. There are 20 cats priced from \$12 to \$15 and 20 sacks priced from 10 cents to \$1 for sale (all prices are different). Prove that each of two boys, John and Peter, can buy a cat in a sack by paying the same amount. of money.

Problem 1.17. Let p and q be two consecutive odd prime numbers. Prove that p + q is a product of at least two positive integers greater than 1 (not necessarily different).

Problem 1.18. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

Problem 1.19. Find all triples (x, y, z) of positive integers satisfying the system of equations:

$$\begin{cases} x^2 = 2(y+z) \\ x^6 = y^6 + z^6 + 31(y^2+z^2) \end{cases}$$

Problem 1.20. In the figure below, you see three half-circles. The circle C is tangent to two of the half-circles and to the line $\overline{\text{PQ}}$ perpendicular to the diameter $\overline{\text{AB}}$. The area of the shaded region is 39π, and the area of the circle C is 9π. Find the length of the diameter $\overline{\text{AB}}$.